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Peierls Proof of Spontaneous Magnetization in a Two-Dimensional Ising Ferromagnet*

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A few minor modifications are made in the Peierls argument that a two-dimensional Ising ferromagnet possesses a spontaneous moment at sufficiently low temperatures, in order to make the proof quite rigorous.

I. INTRODUCTION

THE two-dimensional Ising ferromagnet—we here restrict discussion to the simple square lattice—has been the subject of numerous theoretical investigations.[1] The thermodynamic properties in the absence of a magnetic field are known exactly,[2] and the spontaneous magnetization below the critical temperature has been computed.[3] Some years before the exact results were available, Peierls[4] published a simple argument showing that at sufficiently low temperatures a spontaneous magnetization must exist. This argument is valuable because of the insight it gives into why the two-dimensional lattice, in contrast to the one-dimensional linear chain, shows a phase transition. It is also easily extended, for example, to the three-dimensional cubic lattice. No one doubts that in this last case a spontaneous magnetization occurs; nevertheless, exact results are not yet available.

The only difficulty with the Peierls argument is that, as it stands, it is not quite rigorous[5] (see the Appendix). This defect we now attempt to remedy.

II. CONFIGURATIONS AND BORDERS

Consider a square lattice measuring $\sqrt{N}\times\sqrt{N}$ unit cells and containing N spins. Each spin may be either up or down, leading to a total of 2^N possible configurations. In a particular configuration we may, following Peierls, draw a series of borders[6] between the + (up) spins and the − (down) spins, as illustrated in Fig. 1. To avoid ambiguity, we assume each border is drawn in a particular sense so as to keep the − spins to the right and the + spins to the left. Where this prescription leads to two possibilities, the border always bends to the right (see Fig. 1). Thus, no two borders ever cross. Note that some borders are closed curves, whereas others begin and end on the boundaries of the square. Two borders containing the same line segments but having opposite orientation (they cannot both occur in the same configuration) are considered as distinct.

The energy E of a configuration shall be taken as $2J$ (>0) times the number of pairs of neighboring spins which are oppositely directed, thus $2J$ times the total length (with lattice constant$=1$) of all the borders occurring in the configuration. The probability P_j of occurrence of the jth configuration is taken as

$$P_j=e^{-E_j/kT}\left[\sum_l e^{-E_l/kT}\right]^{-1}, \tag{1}$$

where the sum extends over all configurations.

III. SPONTANEOUS MAGNETIZATION

For any configuration, the average magnetization per spin shall be defined as

$$M=(N_+-N_-)/2N, \tag{2}$$

where N_+ and N_- are the number of + and − spins in the configuration. It is clear from symmetry that the average of M taken over all configurations weighted with the probability (1) is zero. On the other hand, the average of $|M|$ will be greater than zero, and we shall, for our present purposes, define the spontaneous magnetization by

$$M_0=\lim_{N\to\infty}\langle|M|\rangle, \tag{3}$$

where the angular brackets denote the thermal average.

In fact, it is not at all easy to show that the limit (3) exists and, if it exists, that it is independent of boundary conditions. We shall be content with proving that, at a

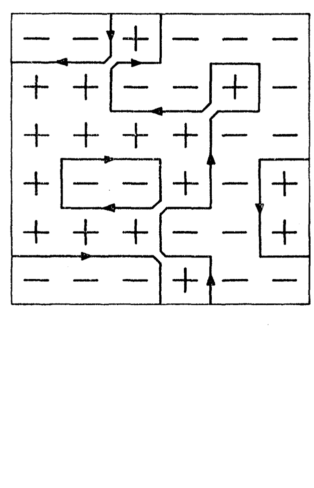

FIG. 1. Example of a configuration showing borders separating + and − spins.

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[1] See, in particular, the review articles by G. F. Newell and E. W. Montroll, Rev. Mod. Phys. **25**, 353 (1953); C. Domb, Phil. Mag. Suppl. **9**, 149 (1960).

[2] L. Onsager, Phys. Rev. **65**, 117 (1944).

[3] C. N. Yang, Phys. Rev. **85**, 808 (1952).

[4] R. Peierls, Proc. Cambridge Phil. Soc. **32**, 477 (1936). A similar argument is found in G. H. Wannier, *Elements of Solid State Theory* (Cambridge University Press, Cambridge, England, 1959), p. 105.

[5] This was pointed out to the author by N. G. van Kampen and M. E. Fisher.

[6] Peierls (Ref. 4) calls these "boundaries," but we shall reserve the term "boundary" for one of the four edges of the $\sqrt{N}\times\sqrt{N}$ square.

sufficiently low temperature,[7]

$$\langle |M| \rangle \geq M_1 > 0, \quad (4)$$

independent of N.

(A) As a preliminary calculation, not without interest in itself, we shall obtain a lower bound to $\langle M \rangle$ in the case where the $4\sqrt{N}-4$ spins on the boundary of the square are all $+$. That is, only configurations of this type, which we shall say belong to class $\mathcal{P}$, are included in the thermal average.

For configurations in class $\mathcal{P}$, all the borders between $+$ and $-$ spins are closed curves, and every $-$ spin is located inside at least one closed border. A border of length b encloses at most $b^2/16$ spins; hence, the number of $-$ spins, N_-, is less than

$$N_- \leq \sum_{b=4,6,8\cdots} (b^2/16) \sum_{i=1}^{m(b)} X_b{}^{(i)}, \quad (5)$$

where $X_b{}^{(i)}$ is 1 if the ith border of length b occurs in a configuration and 0 otherwise. The quantity $m(b)$, the number of possible borders of length b, is bounded by[4,8]

$$m(b) \leq 4N3^b/3b. \quad (6)$$

Let us focus our attention on the lth border of length b, which we denote by B. The probability of occurrence of this border, that is, the thermal average of $X_b{}^{(l)}$, is

$$\langle X_b{}^{(l)} \rangle = (\sum_i{}' e^{-E_i/kT})(\sum_i e^{-E_i/kT})^{-1}, \quad (7)$$

where the sum in the numerator is restricted to those configurations of class $\mathcal{P}$ in which the border B occurs, and the denominator is a sum over all configurations in class $\mathcal{P}$.

If C is a configuration in which the border B occurs, let C^* be the corresponding configuration in which *every* spin inside B is reversed (the $+$ spins becoming $-$ and vice versa). C^* belongs to $\mathcal{P}$ if C belongs to $\mathcal{P}$, and their energies are related by

$$E_C = E_{C^*} + 2Jb. \quad (8)$$

The denominator on the right side of (7) will only decrease if we restrict the sum to those configurations C^* obtained from configurations in the numerator by reversing all spins inside B. Thus an upper bound on $\langle X_b{}^{(l)} \rangle$ is given by

$$\langle X_b{}^{(l)} \rangle \leq e^{-2Jb/kT}. \quad (9)$$

Now take the thermal average of both sides of (5), using the estimates (6) and (9):

$$\langle N_- \rangle \leq \frac{N}{12} \sum_{b=4,6,8\cdots} b3^b e^{-2Jb/kT} = \frac{N\kappa^4}{6} \frac{2-\kappa^2}{(1-\kappa^2)^2}, \quad (10)$$

provided

$$\kappa = 3e^{-2J/kT} < 1. \quad (11)$$

At $\kappa=\frac{1}{2}$, for example, we have $\langle M \rangle \geq 0.37$, independent of N.

(B) Now let us return to the problem of estimating $\langle |M| \rangle$ without imposing constraints upon the boundary spins. Each border divides all the spins in the square into two sets, those lying to the right and those lying to the left. If the border is closed, one set of spins lies inside the border and one set lies outside. If the border is not closed we shall, for purposes of this calculation, define the smaller of the two sets as lying "inside" the border, and the larger as lying outside. If the two sets contain equal numbers of spins, the set to the right of the border will be said to lie "inside" the border. There are at most $\frac{1}{2}b^2$ spins lying inside a border of length b.

An upper bound to the probability of occurrence of the ith border of length b is again given by (9), since the arguments following Eq. (7) may be repeated almost word for word. (The class $\mathcal{P}$ now becomes the class of all configurations.)

We divide the configurations into two classes.

Class $\mathcal{A}$: All minus spins lie inside some border.[9]

Class $\mathcal{B}$: There is at least one minus spin which lies outside all borders.

Any $+$ spin in a configuration C belonging to class $\mathcal{B}$ must lie on the opposite side of some border from at least one $-$ spin lying outside all borders. Hence, all $+$ spins in C lie inside at least one border.

Let P_j be given by (1) and let $\sum[\mathcal{A}]$, $\sum[\mathcal{B}]$ denote sums over all configurations in classes $\mathcal{A}$ and $\mathcal{B}$, respectively.

$$\begin{aligned}\langle |M| \rangle &= \sum_j |M|_j P_j \geq \sum[\mathcal{A}] M_j P_j - \sum[\mathcal{B}] M_j P_j \\ &= \tfrac{1}{2} - N^{-1}[\sum[\mathcal{A}](N_-)_j P_j + \sum[\mathcal{B}](N_+)_j P_j] \quad (12)\end{aligned}$$

where by M_j, $(N_-)_j$, $(N_+)_j$ we denote the values of these quantities in the jth configuration.

For configurations in class $\mathcal{A}$, all minus spins lie inside some border; hence, the inequality (5) holds with $b^2/16$ replaced by $b^2/2$:

$$N_- \leq \frac{1}{2} \sum_{b=4,6,8\cdots} b^2 \sum_{i=1}^{m(b)} X_b{}^{(i)}. \quad (13)$$

The same inequality holds for N_+ for all configurations in class $\mathcal{B}$. Thus (note that $\mathcal{A}$ and $\mathcal{B}$ are disjoint sets):

$$\begin{aligned}[\sum[\mathcal{A}](N_-)_j P_j + \sum[\mathcal{B}](N_+)_j P_j] &\leq \frac{1}{2} \sum_{b=4,6,8\cdots} b^2 \sum_{i=1}^{m(b)} \\ &\times \langle X_b{}^{(i)} \rangle \leq (4/3) N\kappa^4 (2-\kappa^2)(1-\kappa^2)^{-2}, \quad (14)\end{aligned}$$

with κ (assumed to be <1) given by (11).[10] Thus,

[7] Note that $\langle |M| \rangle$ is less than or equal to another measure of the spontaneous magnetization, $(\langle M^2 \rangle)^{1/2}$, a quantity closely related to the long-range order. See T. D. Schultz, D. C. Mattis, and E. H. Lieb, Rev. Mod. Phys. **36**, 856 (1964).

[8] B. L. van der Waerden, Z. Physik **118**, 473 (1941).

[9] Note that it is quite possible for the number of minus spins in a configuration in class $\mathcal{A}$ to exceed $\frac{1}{2}N$ when N is large.

[10] We could, by taking special account of those borders which begin and end on the boundary of the square, obtain an estimate (14) just as good as (10) in the limit of large N.

provided the temperature is low enough (κ is small enough), we obtain a lower bound of the form (4), independent of N.

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APPENDIX

Peierls gives an expression (3) in his original paper[4]:

$$(4\lambda)^L(1-4\lambda) \quad [\lambda=e^{-2J/kT}<\tfrac{1}{4}], \tag{A1}$$

which is supposed to be an upper bound on the numbers of borders of length L passing through a given point. The reasoning leading to this result is, unfortunately, rather obscure; the result itself is incorrect, at least near $\lambda=\frac{1}{4}$. Since no border in a square containing N spins may have a length exceeding $4N$, it is clear than when λ is sufficiently close to $\frac{1}{4}$, (A1) implies that the probability of *any* border passing through a point is arbitrarily small. This cannot be correct.

The derivation of a similar expression at the top of p. 106 of Wannier's book[4] is unclear and the expression incorrect. When the temperature is sufficiently high the denominator diverges, and the probability of finding any border of finite length goes to zero.

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Thermal Conductivity and Electrical Resistivity of Terbium Between 5 and 300°K

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The thermal conductivity λ of polycrystalline terbium has been studied as a function of temperature T between 5 and 300°K. The λ-versus-T curve exhibits a maximum of 0.205 W cm^{-1} °K^{-1} at 23°K. The antiferromagnetic-paramagnetic transition, $T_{\text{A-P}}$, causes an anomaly in the thermal conductivity at about 225°K. The ferromagnetic-antiferromagnetic transformation, $T_{\text{F-A}}$, because of the narrow antiferromagnetic region, is not observable from the λ versus-T curve. According to the electrical resistivity data, $T_{\text{F-A}}=219\pm1$°K and $T_{\text{A-P}}=230\pm1$°K. The Lorenz function, calculated from the measured thermal conductivity and electrical resistivity values on the same sample, indicates that heat is transported mainly by electrons, with possible additional transport by phonons and magnons. The intrinsic electrical resistivity between 5 and 20°K is proportional to $T^{4.19\pm0.06}$.

INTRODUCTION

RECENTLY, we have initiated thermal conductivity measurements on the rare-earth metals from about 5 to 300°K in order to enlarge the knowledge of heat transport in substances exhibiting various magnetic states. Up to the present time such studies have been completed on dysprosium[1] and gadolinium.[2] In this paper we present our measurements on terbium with a discussion of their significance.

EXPERIMENTAL CONSIDERATIONS

The initial stock of terbium was obtained from Research Chemicals. This material was arc-melted for about 12 min. The partial analysis of the original terbium, provided by the supplier, is summarized in Table I. The electrical resistivity at 4.2°K before arc-melting was found to be 7.01 $\mu\Omega$ cm. After the melting a rod of diameter 0.572 cm and length about 8 cm was cut from the ingot. This rod was swaged to a diameter of 0.476 cm. A section of this material, about 6 cm long, was wrapped in a tantalum foil, sealed into a silica capsule evacuated to 10^{-5} mm Hg, and heat treated at 790°K for 40 h. After this procedure, the specimen was allowed to cool to room temperature in about 3 h. The electrical resistivity at 4.2°K of this specimen was 4.85 $\mu\Omega$ cm.

The thermal conductivity measurements, obtained with increasing temperatures from 5°K, were made using the apparatus described in detail elsewhere.[1] The electrical resistivities on the same sample with the thermal contacts used as potential contacts were made with the equipment briefly discussed before.[3]

TABLE I. Partial analysis of the initial terbium stock.

Impurities	Amount (weight %)
O_2	0.08
Y	0.06
Ca	0.01
Si	0.01
Mg	0.003

* Deceased 26 March 1964.
[1] R. V. Colvin and S. Arajs, Phys. Rev. **133**, A1076 (1964).
[2] S. Arajs and R. V. Colvin, J. Appl. Phys. **35**, 1043 (1964).
[3] S. Arajs, R. V. Colvin, and M. J. Marcinkowski, J. Less-Common Metals 4, 46 (1962).